Notes on Recurrent Problems – Concrete Mathematics Chapter 1

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1 Tower of Hanoi

1.1 Key points

- We *hope* to find the following by playing with the problem:

$$T_n \leq 2T_{n-1} + 1 \quad \text{for } n > 0.$$

- We must move $n-1$ disks before being able to move n^{th} disk

$$\therefore T_n \geq 2T_{n-1} + 1 \quad \text{for } n > 0.$$

- By looking at many cases n, we *might guess* at closed-form:

$$T_n = 2^n - 1, \quad \text{for } n \geq 0. \tag{1}$$

1.2 Proof of closed-form

Proof. We proceed using induction.

Base Case: $n = 0$. The equation is simply

$$\begin{aligned} n &= 0 \\ \therefore T_0 &= 2^0 - 1 \\ &= 0 \end{aligned}$$

Inductive Hypothesis: Assume (1) is true for $n - 1$.

$$\begin{aligned} T_n &= 2T_{n-1} + 1 \\ &= 2(2^{n-1} - 1) + 1 \\ &= 2^n - 2 + 1 \\ \therefore T_n &= 2^n - 1 \end{aligned}$$

By induction, we have shown that the hypothesis holds for $n + 1$ □

The recurrence can be solved using other methods. The one shown in the chapter (of adding 1 to each side etc) leaves some open questions:

- How do we know to add 1 on each side?
- How do we know to represent $U_n = T_{n-1} + 1$?

2 Lines in the Plane

Prove that the closed-form for the recurrence is

$$L_n = \frac{n(n-1)}{2} + 1, \quad \text{for } n \geq 0. \tag{2}$$

Proof. We proceed using induction.

Base Case: $n = 0$.

$$\begin{aligned} L_0 &= \frac{0(0+1)}{2} + 1 \\ \therefore L_0 &= 1 \end{aligned}$$

Inductive Hypothesis: Assume (2) to be true for $n - 1$
Inductive Step: From recurrence

$$\begin{aligned} L_n &= L_{n-1} + n \\ &= \left(\frac{n(n-1)}{2} + 1\right) + n \\ &= \frac{n^2}{2} - \frac{n}{2} + 1 + n \\ &= \frac{n^2}{2} + \frac{n}{2} + 1 \\ &= \frac{n(n-1)}{2} + 1 \end{aligned}$$

$$\begin{aligned} L_n &= L_{n-1} + n \\ &= \frac{(n-1)(n-1+1)}{2} + n \\ &= \left(\frac{n(n-1)}{2} + 1\right) + n \\ &= \frac{n(n-1)}{2} + \frac{2}{2} + \frac{2n}{2} \\ &= \frac{2n + 2 + n(n-1)}{2} \\ &= \frac{2n + 2 + n^2 - n}{2} \\ &= \frac{n^2 + n + 2}{2} \\ \therefore L_n &= \frac{n(n-1)}{2} \end{aligned}$$

Thus, by induction we prove that the hypothesis holds for $n + 1$ □

3 The Josephus Problem

We have to tackle even n and odd n separately. We set $2n$ people in even case because integers multiplied by 2 always result in even number. Adding 1 to an even integer always results in odd, hence we set $2n + 1$ for the other case. It might take some effort to see that $J(5 \cdot 2^m) = 2^{m+1} + 1$.

3.1 Proving the closed form

We want to prove the following for both — odd and even — cases.

$$J(2^m + \ell) = 2\ell + 1 \tag{3}$$

Proof. We proceed with induction on two separate cases.

Base Case: $n = 1$.

$$\therefore m = 0, \quad \ell = 0.$$

$$\begin{aligned} J(2^m + \ell) &= 2^0 + 0 \\ &= 1 \end{aligned}$$

Inductive Step: Suppose that for all n such that $n = 2^k + r$, we have $J(n) = J(2^k + r) = 2r + 1$

1. Suppose $m > 0$ and $n = 2^m + r = 2\ell$. That is, n is even. From recurrence:

$$\begin{aligned} J(2n) &= 2J(n) - 1 \\ \therefore J(n) &= 2J\left(\frac{n}{2}\right) - 1 \\ \therefore J(2^m + \ell) &= 2J\left(\frac{2^{m-1} + \ell}{2}\right) - 1 \end{aligned}$$

   Now, assuming the hypothesis to be true for smaller n:

$$J\left(\frac{2^{m-1} + \ell}{2}\right) = \frac{2\ell}{2} + 1$$

$$\begin{aligned} \therefore J(2^m + \ell) &= 2\left(\frac{2\ell}{2} + 1\right) - 1 \\ &= \frac{4\ell}{2} + 2 - 1 \\ \therefore J(2^m + \ell) &= 2\ell + 1 \end{aligned}$$

2. Suppose $m > 0$ and $n = 2^m + r = 2\ell + 1$. That is, n is odd. Similar to the even case,

$$J(n) = 2J\left(\frac{n}{2} - 1\right) + 1$$

Assuming the hypothesis to be true, we get

$$J(2^{m-1} + \frac{\ell}{2} - 1) = 2J(2^{m-1} + \frac{\ell}{2} - 1) + 1$$

$$\begin{aligned}
\therefore J(n) &= 2\left(\frac{2\ell}{2} + 1 - 1\right) + 1 \\
&= 2\left(\frac{2\ell}{2}\right) + 1 \\
&= 2\ell + 1
\end{aligned}$$

Thus by induction we prove the closed-form for both odd and even cases of n.

□

3.2 Checking where $J(n) = \frac{n}{2}$ works

$$\begin{aligned}
J(n) &= \frac{n}{2} \\
\therefore 2\ell + 1 &= \frac{2^m + \ell}{2} \\
\therefore 2^m + \ell &= 2(2\ell + 1) \\
\therefore 2^m &= 4\ell - \ell + 2 \\
\therefore 2^m &= 3\ell + 2 \\
\therefore 3\ell &= 2^m - 2 \\
\therefore \ell &= \frac{1}{3}(2^m - 2)
\end{aligned}$$

3.3 Generalisation of Josephus Problem

We convert constants in the recurrence into variables:

$$\begin{aligned}
f(1) &= \alpha, \\
f(2n) &= 2f(n) + \beta, \quad \text{for } n \geq 1; \\
f(2n+1) &= 2f(n) + \gamma, \quad \text{for } n \geq 1.
\end{aligned} \tag{4}$$

Equations in (1.12) of the chapter tells us that $f(n)$ can be written as:

$$f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma \tag{5}$$

3.3.1 Repertoire Method

We find settings for general parameters (in our case α, β, γ) for which we know the solution. This gives us *repertoire* of special cases that we can solve. Usually, we need as many special cases as there are parameters.

$$\begin{aligned}
A(n) &= 2^m; \\
B(n) &= 2^m - 1 - \ell; \\
C(n) &= \ell.
\end{aligned} \tag{6}$$

Proof. We proceed with repertoire method.

1. Special case $\alpha = 1, \beta = \gamma = 0$.

$$f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$$

$$\begin{aligned} f(1) &= A(1)\cdot 1 + B(1)\cdot 0 + C(1)\cdot 0 \\ \therefore f(1) &= A(1) \\ \therefore A(1) &= 1 \quad \text{from recurrence (4).} \end{aligned}$$

$$\begin{aligned} f(2n) &= 2A(n)\alpha + B(n)\beta + C(n)\gamma \\ &= 2A(n) + 1\cdot\beta \\ &= 2A(n) \end{aligned}$$

$$\begin{aligned} f(2n+1) &= 2A(n)\alpha + B(n)\beta + C(n)\gamma \\ &= 2A(n)\cdot 1 + 0\cdot 0 + 1\cdot 0 \\ &= 2A(n) \end{aligned}$$

Proof. Let's show $A(n) = 2^m$. We proceed with proof by induction to show:

$$A(2^m + \ell) = 2^m \tag{7}$$

Base Case: $n = 1$.

$$\therefore m = 0, \quad \ell = 0.$$

$$\begin{aligned} A(2^0 + 0) &= 1 \\ \therefore A(1) &= 1 \end{aligned}$$

Inductive Step: Assume the hypothesis (7) to be true for smaller n like we did for recurrence solution in §3.3.1

$$\begin{aligned} A(2n) &= 2A(n) \\ \therefore A(n) &= 2A\left(\frac{n}{2}\right) \\ \therefore A(n) &= 2A\left(\frac{2^{m-1}+\ell}{2}\right) \end{aligned}$$

Assuming the hypothesis (7) for $n-1$

$$A(2^{m-1} + \frac{\ell}{2}) = 2^{m-1}$$

$$\therefore A(n) = 2 \cdot 2^{m-1}$$
$$\therefore A(n) = 2^m$$

Similarly,

$$\begin{aligned} A(2n+1) &= 2A(n) \\ \therefore A(n) &= 2A\left(\frac{n}{2} - 1\right) \\ \therefore A(n) &= 2A\left(\frac{2^{m-1}+\ell}{2} - 1\right) \\ &= 2 \cdot 2^{m-1} \\ \therefore A(n) &= 2^m \end{aligned}$$

Thus, we show by induction that $A(n) = 2^m$.

□

2. Are there any constants (α, β, γ) that give (4)?

   Take $f(n) = 1$. Since f always gives 1, it is called a "constant function".

$$\begin{aligned} f(n) &= 1 \\ \therefore \alpha &= 1 \\ \text{And,} & \\ 1 &= 2 \cdot f(n) + \beta \\ \therefore 1 &= 2 \cdot 1 + \beta \\ \therefore \beta &= -1 \\ \text{And,} & \\ 1 &= 2 \cdot f(n) + \gamma \\ \therefore 1 &= 2 + \gamma \\ \therefore \gamma &= -1 \end{aligned}$$

   Plugging these values of α, β, γ in (5), we get

$$\begin{aligned} f(n) &= A(n) \cdot 1 + B(n) \cdot (-1) + C(n) \cdot (-1) \\ &= A(n) - B(n) - C(n) \end{aligned}$$

$$\begin{aligned} B(n) &= A(n) - 1 - C(n) \\ \therefore B(n) &= 2^m - 1 - \ell \end{aligned}$$

3. Let's set $f(n) = n$

$$
\begin{aligned}
1 &= \alpha \\
&\text{And,} \\
2n &= 2f(n) + \beta \\
\therefore 2n &= 2n + \beta \\
\therefore \beta &= 0 \\
&\text{And,} \\
2n + 1 &= 2f(n) + \gamma \\
\therefore 2n + 1 &= 2n + \gamma \\
\therefore \gamma &= 1
\end{aligned}
$$

Plugging these values of α, β, γ into (5), we get:

$$
\begin{aligned}
f(n) &= A(n)\alpha + B(n)\beta + C(n)\gamma \\
\therefore n &= A(n) \cdot 1 + B(n) \cdot 0 + C(n) \cdot 1 \\
\therefore n &= A(n) + C(n)
\end{aligned}
$$

With the three cases, we now have

$$
\begin{aligned}
A(n) &= 2^m \\
A(n) - B(n) - C(n) &= 1 \\
A(n) + C(n) &= n
\end{aligned}
$$

Now,

$$
\begin{aligned}
C(n) &= n - A(n) \\
&= n - 2^m \\
\therefore C(n) &= \ell \\
&\text{And,} \\
B(n) &= A(n) - 1 - C(n) \\
\therefore B(n) &= 2^m - 1 - \ell
\end{aligned}
$$

Thus we have our hypothesis from (6) proven. □

3.3.2 Checking if bit-shift property holds

$$
f((b_m b_{m-1} \dots b_1 b_0)_2) = (\alpha \beta_{b_{m-1}} \beta_{b_{m-2}} \dots \beta_{b_1} \beta_{b_0})_2 \tag{8}
$$

Equation (8) is not strictly binary radix. Instead of allowing 0 and 1 values for β, we are allowing *any* values. This is because our parameters (α, β, γ) are general and can take any value.

For $n = 100$ in Josephus Problem, we have $\alpha = 1, \beta = -1, \gamma = 1$. Also note that $\beta_{b_{m-1}} = \beta_1 = 1$, $\beta_{b_{m-3}} = \beta_0 = -1$ and so on. The cyclic left bit-shift propery from earlier holds in the generalised form as well:

$$
\begin{aligned}
100 &= (1100100)_2 \\
73 &= (1001001)_2
\end{aligned}
$$